Two uncommon examples of the relativistic motion of an accelerated point particle

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Abstract

In this script we introduce two small yet uncommon examples on the motion of an accelerated point particle in the special theory of relativity: First, the motion in an oscillating potential, $-\mathcal{E}\sin(\omega x^0)$, and second, the problem of the relativistic harmonic oscillator. The aim of this script is threefold: First, illustrate the Lagrangian and Hamiltonian formalism for the relativistic mechanics of a point particle by means of two simple problems. Second, give a counter-example against the sometimes encountered belief that accelerated motion could not be handled by the special theory of relativity. Third, calculate the time-dilation of the twin-paradox in an example problem with steady motion and velocity.

1 Introduction

This script is concerned with the accelerated motion of a point particle with charge e under the influence of a four-vector potential A. We will present the general equations of motion and analytically derive the solution for two simple special cases of A.

Three basic motivations lead to the writing of this script: First, illustrating the Lagrangian and Hamiltonian formalisms as they apply to special relativistic point particle mechanics (although not much of the formalism would have to be altered for a general relativistic application). The formalism is then applied to the case where the four-vector potential is an oscillating potential, $eA = -e\mathcal{E}\sin(\omega x^0)\, e_1$, and a harmonic oscillator potential, $eA = m\omega^2(x^1)^2/2\, e_0$. These special cases have the advantage that the equation of motion may be solved in closed form. The examples are simple and yet not so simple that the calculations become trivial.

The second motivation is the still existing belief that the special theory of relativity is unable to describe accelerated motion. Despite its obvious falseness this belief can be encountered now and then and this script delivers excellent counter-examples.

Finally, the third point is concerned with the calculation of the time-dilation in the twin-paradox. Usual sample calculations on the amount of time-dilation involve unrealistic motion with instantaneous change of velocity. A common example would be the separation of motion into two phases, I and II, of equal duration and considering motion with constant velocity v in phase I and $-v$ in phase II. Both problems presented in this script involve periodic motion, i.e. the particle returns to its starting point after a finite period of time. Hence they serve as far more realistic sample cases to calculate the amount of time-dilation as they involve steady changes of position and velocity and do not require a separation into different phases.

One should note, however, that the aforementioned examples are still not fully realistic but can serve only as illustrations for the underlying physical principles. While the electromagnetic field that follows from the two four-vector potentials might indeed be realized by an appropriate experimental setup, the accelerated motion of the charged particle would deviate from our solutions due to energy-loss by the emission of radiation. This energy-loss is ignored in the following calculations, however, because we would like to concentrate on problems that are analytically solvable. Also, while radiative energy-loss would change the particle's motion, especially in the extreme relativistic scenarios, it could in no way cause the disappearance of relativistic effects such as time-dilation.

We begin in section 2 with the introduction of the general formalism for the mechanics of a relativistic point particle in an external field. This formalism is then applied to the aforementioned oscillating potential in section 3 and to the relativistic harmonic oscillator in section 4. A short summary is presented in section 5.

2 The formalism

In this section we will give an introduction to the Lagrangian and Hamiltonian formalism in order to describe the motion of a relativistic charged particle in an external four-vector potential A. First, however, we will clarify the notation.

We are concerned with special relativistic problems, i.e. it is appropriate to describe spacetime by the four-dimensional Minkowski space $\mathcal{M}$ with metric g. In Minkowski space it is possible to define a coordinate frame $\mathcal{C}$ with coordinates (x^μ), $\mu = \{0, 1, 2, 3\}$, such that the spacetime metric g is given by the Minkowski metric η,

$$g \to \eta = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2. \tag{1}$$

The coordinate frame $\mathcal{C}$ coincides with the rest-frame of an inertial observer. Let us denote basis vectors and one-forms induced by $\mathcal{C}$ as (e_μ) and (e^μ), respectively,

$$e_\mu = \partial_\mu, \qquad\qquad e^\mu = dx^\mu. \tag{2}$$

For a spacetime vector v and one-form ω we write

$$v = v^\mu e_\mu, \qquad\qquad \omega = \omega_\mu e^\mu, \tag{3}$$

respectively, where we have made use of Einstein's summation convention for contracted upper and lower indices,

$$v^\mu \omega_\mu := \sum_{\mu=0}^{3} v^\mu \omega_\mu. \tag{4}$$

Every vector v defines a one-form $\underline{v}$ via

$$\underline{v} = v_\mu e^\mu, \qquad\qquad v_\mu = g_{\mu\nu} v^\nu. \tag{5}$$

The scalar product between two vectors v and u is denoted by a dot-product,

$$\begin{aligned} v \cdot u &= \underline{v}(u) = \underline{u}(v) \\ &= g_{\mu\nu} u^\mu v^\nu = \eta_{\mu\nu} u^\mu v^\nu \\ &= -u^0 v^0 + u^1 v^1 + u^2 v^2 + u^3 v^3. \end{aligned} \tag{6}$$

The scalar product of a vector v with itself is simply denoted by v^2. The Minkowski metric η in eq. (1) can be used to define pseudo rotations, the well known Lorentz transformations. The transformations matrix $\Lambda = (\Lambda^\mu{}_\nu)$ of such a Lorentz transformation satisfies

$$\eta(\Lambda, \Lambda) = (\eta_{\alpha\beta} \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu) = (\eta_{\mu\nu}) = \eta. \tag{7}$$

The set of Lorentz transformations forms a group as can be easily proven. From the definition (6) we can see that the scalar product is invariant under Lorentz transformations,

$$v \cdot u = v' \cdot u', \qquad v' = \Lambda \cdot v, \qquad u' = \Lambda \cdot u. \tag{8}$$

The action $S[x_f, x_i]$ for a relativistic point particle's motion between the events x_i and x_f in spacetime is invariant under Lorentz transformations,

$$S[x'_f, x'_i] = S[x_f, x_i], \qquad x'_{f/i} = \Lambda \cdot x_{f/i}, \qquad {x'_{f/i}}^2 = {x_{f/i}}^2. \tag{9}$$

A possible choice for S, which has the correct non-relativistic limit, reads

$$S[x_f, x_i] = -m \int_{\tau_i}^{\tau_f} d\tau \, \sqrt{|\dot{x}^2|}, \tag{10}$$

where $x(\tau_{f/i}) = x_{f/i}$ and $\dot{x} = dx/d\tau$ with the arbitrary world line parameter τ. Obviously this action is not only Lorentz-invariant but also reparameterization-invariant, i.e. invariant under the action of an arbitrary diffeomorphism Φ on τ,

$$S[x_f, x_i] = -m \int_{\tau_i}^{\tau_f} d\tau \, \sqrt{|\dot{x}^2|} = -m \int_{\tau'_i}^{\tau'_f} d\tau' \, \sqrt{|\dot{x}'|^2}, \tag{11}$$

where $\tau' = \Phi(\tau)$, $x(\tau'_{f/i}) = x_{f/i}$ and $\dot{x}' = dx/d\tau'$. The four-velocity $\dot{x}$ is a time-like vector, i.e. we have $\dot{x}^2 < 0$ in all reference frames.

Let us consider the action of a relativistic point particle with charge e under the action of an external four-vector potential A. In this case the action (10) has to be replaced by

$$S[x_f, x_i] = \int_{\tau_i}^{\tau_f} d\tau \, \left(-m\sqrt{|\dot{x}^2|} + eA(x) \cdot \dot{x} \right), \tag{12}$$

which is also reparameterization-invariant. Writing the action in terms of the Lagrangian L yields

$$\begin{aligned} S[x_f, x_i] &= \int_{\tau_i}^{\tau_f} d\tau \, L(x(\tau), \dot{x}(\tau)), \\ L(x(\tau), \dot{x}(\tau)) &= -m\sqrt{|\dot{x}(\tau)|^2} + eA(x(\tau)) \cdot \dot{x}(\tau). \end{aligned} \tag{13}$$

From eqs. (13) we may derive the equation of motion by requiring the full differential δS of the action to vanish for physical trajectories,

$$\begin{aligned} \delta S &= \int_{\tau_i}^{\tau_f} d\tau \, \left(\delta x \cdot \partial_x L + \delta \dot{x} \cdot \partial_{\dot{x}} L \right) \\ &= \delta x \cdot \partial_{\dot{x}} L(x(\tau), \dot{x}(\tau)) \Big|_{\tau_i}^{\tau_f} + \int_{\tau_i}^{\tau_f} d\tau \, \delta x \cdot \left(\partial_x L - \frac{d(\partial_{\dot{x}} L)}{d\tau} \right). \end{aligned} \tag{14}$$

Going from the first to the second line, we have performed a partial integration. The variation is set to vanish at the endpoints $\delta x(\tau_f) = \delta x(\tau_i) = 0$ so that the first term drops out. The Euler-Lagrange equations of motion for the relativistic point particle then follow as

$$\frac{d(\partial_{\dot{x}} L)}{d\tau} = \partial_x L, \tag{15}$$

which resembles the non-relativistic form. Inserting the Lagrangian from eqs. (13), and using $|\dot{x}|^2 = -\dot{x}^2$, yields

$$\begin{aligned} \frac{d}{d\tau}\left(\frac{m\underline{\dot{x}}}{\sqrt{|\dot{x}|^2}} + e\underline{A}\right) &= e\partial_x A \cdot \dot{x} \\ \Leftrightarrow \qquad \frac{d}{d\tau}\frac{m\underline{\dot{x}}}{\sqrt{|\dot{x}|^2}} &= e\left(\partial_x A \cdot \dot{x} - (\dot{x}\cdot\partial_x)\underline{A}\right) \\ &= eF\cdot\dot{x}, \end{aligned} \tag{16}$$

where we have introduced the electromagnetic field strength tensor,

$$F = (F_{\mu\nu}) = (\partial_\mu A_\nu - \partial_\nu A_\mu). \tag{17}$$

The equations of motion of the Lagrangian formalism (16) are second order differential equations in the variable x. In order to obtain the Hamiltonian form of the equations of motion we have to calculate the canonical momenta p,

$$\underline{p} = \partial_{\dot{x}} L(x,\dot{x}) = \frac{m\underline{\dot{x}}}{\sqrt{|\dot{x}^2|}} + e\underline{A}(x). \tag{18}$$

Let us further define modified canonical momenta π via

$$\pi = p - eA = \frac{m\dot{x}}{\sqrt{|\dot{x}^2|}}. \tag{19}$$

Using $\dot{x}^2 < 0$, we obtain for the square of π,

$$\pi^2 = -m^2. \tag{20}$$

Rewriting the action and Lagrangian in eqs. (13), then yields

$$\begin{aligned} S[x_f, x_i] &= \int_{x_i}^{x_f} dx \cdot p, \\ L(x(\tau), \dot{x}(\tau)) &= \dot{x}\cdot p. \end{aligned} \tag{21}$$

From these relations we find that the Hamiltonian of the system vanishes identically,

$$H(x,p) = \dot{x}\cdot p - L(x,\dot{x}) \equiv 0. \tag{22}$$

One should note, however, that eqs. (20) and (22) both follow from eq. (18) which in turn is derived from the Lagrangian and connects position x to momentum p. If we choose to start with the Hamiltonian approach, on the other hand, we have to treat x and p as independent variables. The condition on π imposed by eq. (20) then has to be implemented by a suitable auxiliary function $\mathcal{H}$,

$$S[x_f, x_i] = \int_{x_i}^{x_f} dx \cdot p - \int_{\tau_i}^{\tau_f} d\tau\, \mathcal{H}(\tau, x(\tau), p(\tau)), \tag{23}$$

where $\mathcal{H}$ vanishes if eq. (20) is satisfied. Both the action S, as well as the first term in eq. (23) are reparameterization-invariant so the combination $d\tau\,\mathcal{H}$ must be reparameterization-invariant as well. The condition (20) is also reparameterization-invariant and so we may write

$$\mathcal{H}(\tau, x, p) = \lambda(\tau) H(x, p), \tag{24}$$

where H is itself reparameterization-invariant and vanishes if eq. (20) is satisfied. This is the behavior we expect from a relativistic Hamiltonian.

We find it convenient to define H in a way that resembles the form of a non-relativistic Hamiltonian,

$$H(x, p) = \frac{\pi^2 + m^2}{2m} = \frac{(p - eA(x))^2 + m^2}{2m}. \tag{25}$$

In order to preserve the reparameterization-invariance of the action, the Lagrange multiplier λ has to behave according to

$$d\tau \lambda(\tau) = d\tau' \lambda'(\tau'), \tag{26}$$

under a reparameterization $\tau \to \tau' = \Phi(\tau)$, which in turn implies that

$$\lambda'(\tau') = \frac{d\tau}{d\tau'} \lambda(\tau). \tag{27}$$

We now have obtained the following reparameterization-invariant action

$$\begin{aligned} S[x_f, x_i] &= \int_{x_i}^{x_f} dx \cdot p - \int_{\tau_i}^{\tau_f} d\tau\, \lambda H \\ &= \int_{\tau_i}^{\tau_f} d\tau\, \left(\dot{x} \cdot p - \lambda H\right). \end{aligned} \tag{28}$$

The full differential of the action under variation of the independent variables x, p and λ reads

$$\delta S = \delta x \cdot p \Big|_{\tau_i}^{\tau_f} + \int_{x_i}^{x_f} d\tau\, \Big(-\delta x \cdot (\dot{p} + \lambda \partial_x H) + \delta p \cdot \Big(\dot{x} - \lambda \partial_{\underline{p}} H\Big) - \delta\lambda\, H\Big), \tag{29}$$

where the variation is set to vanish at the endpoints $\delta x(\tau_f) = \delta x(\tau_i) = 0$ as usual, so that the first term drops out. Requiring $\delta S = 0$ for physical trajectories, we obtain the following equations of motion

$$\begin{aligned} \dot{x} &= \lambda \partial_{\underline{p}} H, \\ \dot{\underline{p}} &= -\lambda \partial_x H, \\ H &= 0. \end{aligned} \tag{30}$$

The third line in this set of equations enforces eq. (20) that we had previously obtained in the Lagrangian formalism.

Note, however, that instead of formulating the Hamiltonian equations of motion (30) we could just as well return to the Lagrangian formalism that we started with, keeping the manifest gauge freedom that is represented by the auxiliary field λ,

$$\dot{x} = \lambda \partial_{\underline{p}} H = \lambda \frac{\pi}{m},$$

$$L = \frac{m\dot{x}^2}{2\lambda} + eA \cdot \dot{x} - \frac{\lambda m}{2}. \tag{31}$$

In this way we arrive at a Lagrangian with a kinetic term that looks quite like in the non-relativistic limit, i.e. without the square root from eq. (12),

$$S[x_f, x_i] = \int_{\tau_i}^{\tau_f} d\tau \left(\frac{m\dot{x}^2}{2\lambda} + eA \cdot \dot{x} - \frac{\lambda m}{2} \right). \tag{32}$$

Requiring that the variation of the action with respect to λ vanishes, $\delta S/\delta\lambda = 0$, yields the constraint

$$\dot{x}^2 = -\lambda^2. \tag{33}$$

Since this is not a dynamic equation we could either insert this relation back into eq. (32), which would yield eq. (12) or choose a fixed value for λ, in this way also fixing the world line parameter τ. Let us choose the second alternative and set

$$\lambda \equiv 1 \qquad \Rightarrow \qquad \dot{x}^2 = -1, \tag{34}$$

so that

$$S[x_f, x_i] = -\frac{m(\tau_f - \tau_i)}{2} + \int_{\tau_i}^{\tau_f} d\tau \left(\frac{m\dot{x}^2}{2} + eA(x) \cdot \dot{x} \right). \tag{35}$$

Thus we were able to use the reparameterization-invariance of the action (12) to choose a parameterization such that the action is reminiscent of a non-relativistic point particle's action. With this choice of gauge, the Lagrangian equations of motion (16) simply read

$$m\underline{\ddot{x}} = eF \cdot \dot{x}. \tag{36}$$

In the following two sections we will calculate two simple problems on the relativistic motion of a charged point particle accelerated by an external potential. We will use the Hamiltonian equations of motion (30) in both cases since the calculations turn out to be more convenient in this formalism.

3 An oscillating potential

As a first problem let us consider a point particle in an oscillating potential. We would like to perform the calculation in a coordinate system $\mathcal{C}$ with coordinates (x^0, x^1, x^2, x^3), which coincides with the rest-frame of an inertial observer. Hence the spacetime metric is simply given by eq. (1) in this coordinate system.

Let us now consider the oscillating potential

$$eA(x) = -\mathcal{E} \sin(\omega x^0)\, e_1, \tag{37}$$

where $\mathcal{E}$ is the field strength and ω the cyclic frequency of the oscillation. The Lagrangian (31) and Hamiltonian (25) with this potential read, respectively,

$$L(x, \dot{x}) = \frac{m}{2} \left[-1 - (\dot{x}^0)^2 + (\dot{x}^1)^2 + (\dot{x}^2)^2 + (\dot{x}^3)^2 \right] - \dot{x}^1 \mathcal{E} \sin(\omega x^0),$$

$$H(x,p) = \frac{-(p_0)^2 + (p_1 + \mathcal{E}\sin(\omega x^0))^2 + (p_2)^2 + (p_3)^2 + m^2}{2m}. \tag{38}$$

The Hamiltonian formalism is the more convenient choice for this particular problem. From the Hamiltonian equations of motion (30) we find

$$\begin{aligned}
\dot{x}^0 &= -\frac{p_0}{m}, & \dot{p}_0 &= -\frac{p_1 + \mathcal{E}\sin(\omega x^0)}{m}\omega\mathcal{E}\cos(\omega x^0),\\
\dot{x}^1 &= \frac{p_1 + \mathcal{E}\sin(\omega x^0)}{m}, & \dot{p}_1 &= 0,\\
\dot{x}^2 &= \frac{p_2}{m}, & \dot{p}_2 &= 0,\\
\dot{x}^3 &= \frac{p_3}{m}, & \dot{p}_3 &= 0,
\end{aligned} \tag{39}$$

together with the Hamiltonian constrain, $H = 0$,

$$(p_0)^2 = \big(p_1 + \mathcal{E}\sin(\omega x^0)\big)^2 + (p_2)^2 + (p_3)^2 + m^2. \tag{40}$$

Let us consider the initial conditions

$$x(0) = 0, \qquad \dot{x}(0) = e_0. \tag{41}$$

Inserting these initial conditions into the equations of motion yields

$$x^2 = x^3 = p_1 = p_2 = p_3 \equiv 0, \qquad p_0(0) = -m, \tag{42}$$

as well as

$$\dot{x}^1 = \frac{\mathcal{E}}{m}\sin(\omega x^0), \qquad \dot{p}_0 = -\frac{\omega\mathcal{E}^2}{m}\sin(\omega x^0)\cos(\omega x^0). \tag{43}$$

Using eq. (40) together with the equations of motion we obtain

$$\dot{x}^0 = \sqrt{1 + (\dot{x}^1)^2} = \sqrt{1 + \alpha^2\sin^2(\omega x^0)}, \tag{44}$$

where we have written $\alpha = \mathcal{E}/m$ and made use of the initial condition $\dot{x}^0(0) = 1$. We can easily verify that this expression also satisfies the equation for $\dot{p}_0$. Since both $\dot{x}^0$ and $\dot{x}^1$ are given as functions of x^0 and not τ we should calculate the velocity $v^1 = dx^1/dx^0$ that is measured in the coordinate frame $\mathcal{C}$,

$$v^1 = \frac{dx^1}{dx^0} = \frac{\dot{x}^1}{\dot{x}^0} = \frac{\alpha\sin(\omega x^0)}{\sqrt{1 + \alpha^2\sin^2(\omega x^0)}}. \tag{45}$$

The maximum of the velocity solely depends on α,

$$\left|v^1\right| \geq \frac{\alpha}{\sqrt{1+\alpha^2}}, \tag{46}$$

and hence on the strength $\mathcal{E}$ of the external field as well as the particle mass m. The velocity v^1 is depicted in fig. 1 for different values of α and ω. Note especially how the velocity profile resembles that of a harmonic oscillator for small values of α but approaches a step-function with increasing α.

The trajectory $x^1(x^0)$ is given by the following relation:

$$x^1(x^0) = \int_0^{x^0} dt\, v^1(t) = \int_0^{x^0} \frac{dt\,\alpha\sin(\omega t)}{\sqrt{1 + \alpha^2\sin^2(\omega t)}}. \tag{47}$$

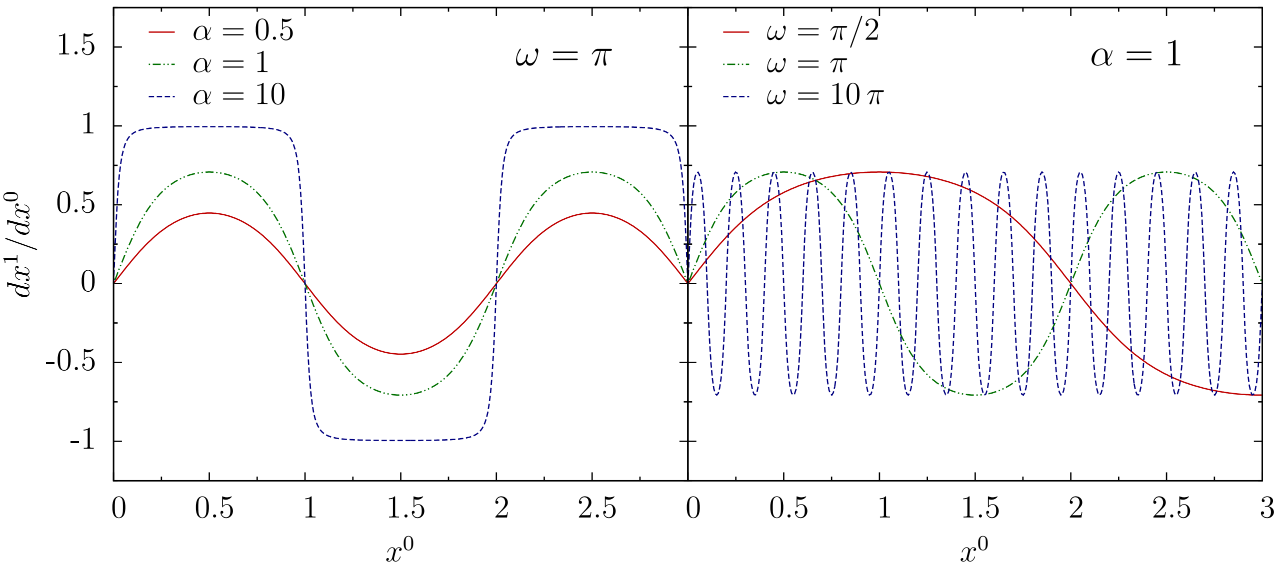

Figure 1: Velocity $v^1 = dx^1/dx^0$ as measured in the coordinate frame $\mathcal{C}$ plotted for varying α, constant ω (left) and constant α, varying ω (right).

In order to solve the integral, let us introduce the variable y,

$$y = \frac{\alpha\cos(\omega t)}{\sqrt{1+\alpha^2}}, \qquad dy = -\frac{dt\,\alpha\omega\sin(\omega t)}{\sqrt{1+\alpha^2}}, \tag{48}$$

so that we obtain

$$\begin{aligned} x^1(x^0) &= -\int_{y(0)}^{y(x^0)} \frac{dy}{\sqrt{1-y^2}} \\ &= \frac{1}{\omega}\left(\arccos(y(x^0)) - \arccos(y(0))\right) \\ &= \frac{1}{\omega}\left(\arccos\left(\frac{\alpha\cos(\omega x^0)}{\sqrt{1+\alpha^2}}\right) - \arccos\left(\frac{\alpha}{\sqrt{1+\alpha^2}}\right)\right). \end{aligned} \tag{49}$$

The trajectory is depicted in fig. 2 for different values of α and ω. The curve of $x^1(x^0)$ oscillates nearly harmonically at small values of α and is forced into a triangle shaped curve at large values of α, with a maximum value of

$$x^1(x^0) \leq \frac{1}{\omega}\left(\pi - 2\arccos\frac{\alpha}{\sqrt{1+\alpha^2}}\right) < \frac{\pi}{\omega}, \tag{50}$$

which is compatible with the frequency ω and the limiting speed of light.

Due to the periodic motion we have a similar situation as in the twin paradox, where one twin is at rest at $x^1 = 0$, say, so that his proper time coincides with the coordinate time x^0. The function $x^1(x^0)$ then describes the trajectory of the other twin who has a different proper time τ. The two twins meet when the function $x^1(x^0)$ passes through zero. We are now interested in the amount of time Δx^0 and $\Delta\tau$ has passed for each of the twins between two consecutive meetings. In order to do this we have to calculate the functional relation $\tau(x^0)$ between the proper time and the coordinate time using eq. (44),

$$\tau(x^0) = \int_0^{x^0} \frac{dt}{\dot{x}^0(t)} = \frac{1}{\omega}\int_0^{\omega x^0} \frac{d\varphi}{\sqrt{1+\alpha^2\sin^2\varphi}}$$

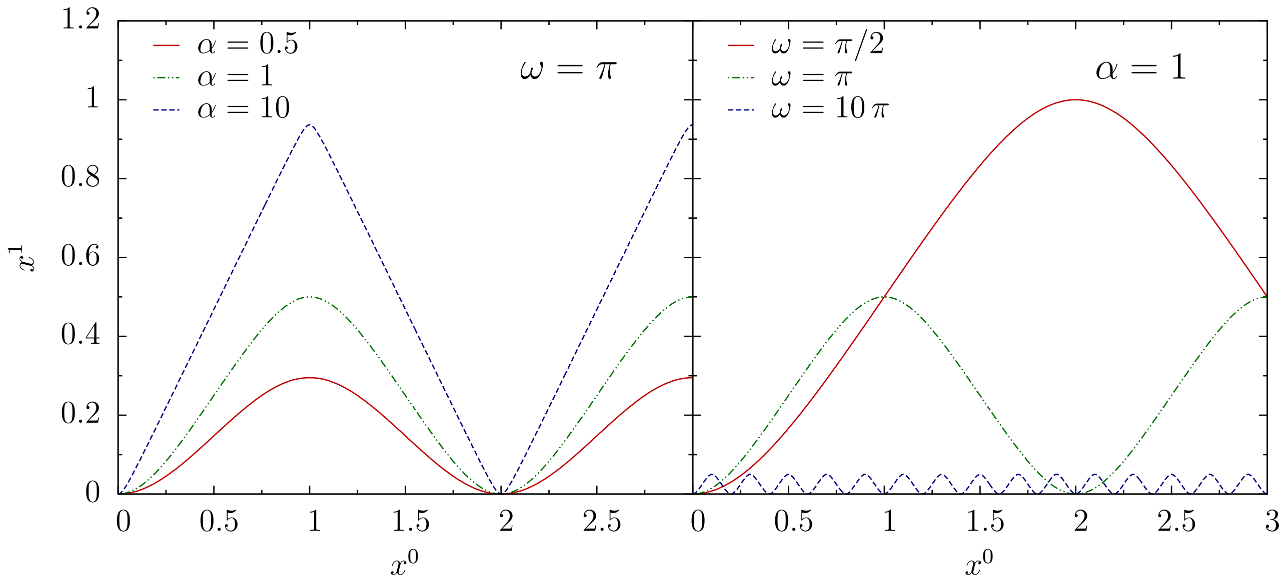

Figure 2: Trajectory $x^1(x^0)$ as measured in the coordinate frame $\mathcal{C}$ plotted for varying α, constant ω (left) and constant α, varying ω (right).

$$= \frac{1}{\omega} F(\omega x^0, -\alpha^2), \tag{51}$$

where the elliptic integral of the first kind $F(\phi, \mu)$ is a function of amplitude ϕ and parameter μ and is defined as,

$$F(\phi, \mu) = \int_0^\phi \frac{dt}{\sqrt{1 - \mu \sin^2 t}}. \tag{52}$$

In the coordinate frame $\mathcal{C}$ the time passed between two zeros of $x^1(x^0)$ amounts to

$$\Delta x^0 = \frac{2\pi}{\omega}. \tag{53}$$

Comparing this to the amount of proper time $\Delta\tau$ that has passed in the rest frame of the particle (the other twin) yields

$$\frac{\Delta\tau}{\Delta x^0} = \frac{2}{\pi\sqrt{1+\alpha^2}} K\left(\frac{\alpha^2}{1+\alpha^2}\right), \tag{54}$$

where $K(\mu) = F(\pi/2, \mu)$ is the complete elliptic integral of the first kind and where we have used the relation

$$K(-\mu) = \frac{1}{\sqrt{1+\mu}} K\left(\frac{\mu}{1+\mu}\right). \tag{55}$$

The ratio $\Delta\tau/\Delta x^0$ is depicted in fig. 3. It is equal to one for $\alpha = 0$ and approaches 0 for $\alpha \to \infty$,

$$1 \leq \frac{\Delta\tau}{\Delta x^0} < 0. \tag{56}$$

Correspondingly, less time has passed between two passages of the accelerated particle (traveling twin) through $x^1 = 0$ in its rest-frame than in the coordinate frame $\mathcal{C}$.

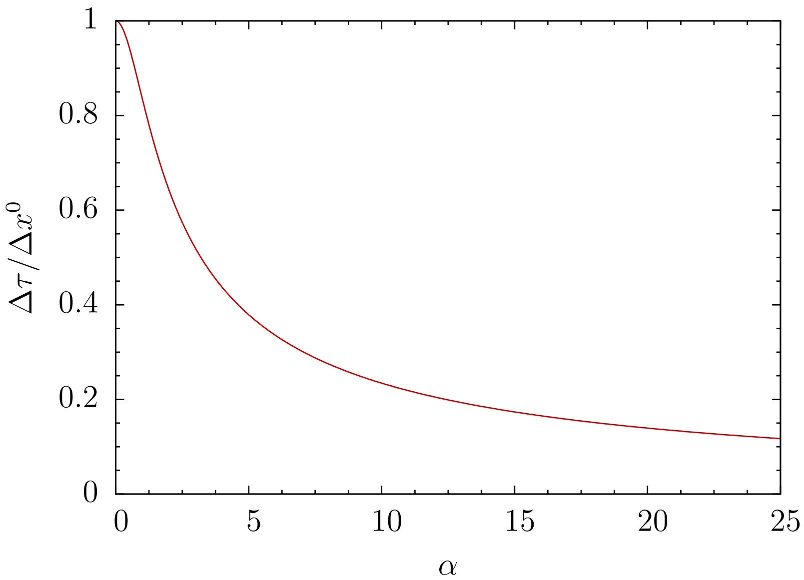

Figure 3: Ratio of the amount of proper time $\Delta\tau$ and coordinate time Δx^0 that has passed between two passages of the particle through $x^1 = 0$ as a function of α.

4 The harmonic oscillator

In this section we will solve the problem of the relativistic harmonic oscillator. Again we choose the coordinate system $\mathcal{C}$ as in section 3. For the harmonic oscillator the four-vector potential in eq. (35) reads

$$eA(x) = \frac{m\omega^2(x^1)^2}{2}\, e_0. \tag{57}$$

where ω is the cyclic frequency of the oscillation. The Lagrangian (31) and Hamiltonian (25) read with this potential, respectively,

$$\begin{aligned} L(x,\dot{x}) &= \frac{m}{2}\left[-1-(\dot{x}^0)^2+(\dot{x}^1)^2+(\dot{x}^2)^2+(\dot{x}^3)^2\right] - \dot{x}^0\frac{m\omega^2(x^1)^2}{2}, \\ H(x,p) &= \frac{-(p_0+m\omega^2(x^1)^2/2)^2+(p_1)^2+(p_2)^2+(p_3)^2+m^2}{2m}. \end{aligned} \tag{58}$$

Again the Hamiltonian formalism turns out to be the more convenient choice for this particular problem. We find from the equations of motion (30) that

$$\begin{aligned} \dot{x}^0 &= -\left(\frac{p_0}{m}+\frac{(\omega x^1)^2}{2}\right), & \dot{p}_0 &= 0, \\ \dot{x}^1 &= \frac{p_1}{m}, & \dot{p}_1 &= \left(\frac{p_0}{m}+\frac{(\omega x^1)^2}{2}\right) m\omega^2 x^1, \\ \dot{x}^2 &= \frac{p_2}{m} & \dot{p}_2 &= 0, \\ \dot{x}^3 &= \frac{p_3}{m} & \dot{p}_3 &= 0 \end{aligned} \tag{59}$$

Further we consider the initial conditions

$$x(0) = de_1, \qquad \dot{x}(0) = e_0. \tag{60}$$

Inserting these initial conditions into the equations of motion yields

$$x^2 = x^3 = p_2 = p_3 \equiv 0, \qquad p_0 \equiv -m\left(1+\frac{(\omega d)^2}{2}\right), \tag{61}$$

as well as

$$\dot{x}^0 = 1 + \frac{\omega^2(d^2 - (x^1)^2)}{2}, \qquad\qquad \dot{p}_1 = -m\omega^2 x^1 \dot{x}^0. \tag{62}$$

Differentiating the equation for $\dot{x}^1$ in eqs. (59) with respect to τ and inserting the new expressions for $\dot{p}_1$ and $\dot{x}^0$ yields a second order ordinary differential equation for x^1,

$$\ddot{x}^1 = -\omega^2\left(1 + \frac{\omega^2 d^2}{2}\right)x^1 + \frac{\omega^4}{2}(x^1)^3. \tag{63}$$

This type of differential equation is satisfied by the Jacobi elliptic functions sn, cn and dn, which are, respectively, defined as

$$\begin{aligned}
\mathrm{sn}(z,\mu) &= \sin(\phi(z,\mu)),\\
\mathrm{cn}(z,\mu) &= \cos(\phi(z,\mu)),\\
\mathrm{dn}(z,\mu) &= \sqrt{1 - \mu\,\mathrm{sn}^2(z,\mu)},
\end{aligned} \tag{64}$$

where the amplitude ϕ is defined as the inverse of the elliptic integral of the first kind

$$z = F(\phi(z,\mu),\mu) \qquad \Leftrightarrow \qquad \phi(z,\mu) = F^{-1}(z,\mu), \tag{65}$$

and where $F(z,\mu)$ is given in eq. (52). The sine and cosine amplitudes sn and cn satisfy, respectively, the relations

$$\begin{aligned}
\mathrm{sn}(2nK(\mu),\mu) &= 0, & \mathrm{cn}(2nK(\mu),\mu) &= (-1)^n,\\
\mathrm{sn}((2n+1)K(\mu),\mu) &= (-1)^n, & \mathrm{cn}((2n+1)K(\mu),\mu) &= 0.
\end{aligned} \tag{66}$$

where $n \in \mathbb{N}$ and K means the complete elliptic integral of the first kind, $K(\mu) = F(\pi/2,\mu)$. The initial conditions (60) for x^1 suggest the following ansatz

$$x^1(\tau) = A\,\mathrm{cn}(B\omega\tau + C, D). \tag{67}$$

Together with the following relation satisfied by the derivative of the cosine amplitude,

$$\frac{d\,\mathrm{cn}}{dz}(z,\mu) = -\,\mathrm{sn}(z,\mu)\,\mathrm{dn}(z,\mu), \tag{68}$$

we obtain

$$\begin{aligned}
x^1(0) &= d = A\,\mathrm{cn}(C,D),\\
\dot{x}^1(0) &= 0 = -AB\omega\,\mathrm{sn}(C,D)\,\mathrm{dn}(C,D).
\end{aligned} \tag{69}$$

Since $A, B \neq 0$, we must have $C = 2nK(D)$, $n \in \mathbb{N}$. Choosing $n = 0$ we obtain $A = d$ and $C = 0$, so that

$$x^1(\tau) = d\,\mathrm{cn}(B\omega\tau, D). \tag{70}$$

Inserting this into eq. (63) yields

$$dB^2\omega^2\frac{d^2\,\mathrm{cn}}{d\tau}(B\omega\tau, D) = -\omega^2\left(1 + 2\alpha^2\right)d\,\mathrm{cn}(B\omega\tau, D) + 2\omega^2\alpha^2 d\,\mathrm{cn}^3(B\omega\tau, D), \tag{71}$$

where we have set $\alpha = \omega d/2$. Using the following relation satisfied by the second derivative of the cosine amplitude,

$$\frac{d^2\,\mathrm{cn}}{d^2 z}(z,\mu) = -(1 - 2\mu)\,\mathrm{cn}(z,\mu) - 2\mu\,\mathrm{cn}^3(z,\mu), \tag{72}$$

we may rewrite this equation as follows,

$$0 = \left(1 + 2\alpha^2 - (1 - 2D)B^2\right) \mathrm{cn}(B\omega\tau, D) - \left(2B^2 D + 2\alpha^2\right) \mathrm{cn}^3(B\omega\tau, D). \tag{73}$$

The expressions in both brackets have to vanish independently and so we obtain

$$B = \pm 1, \qquad\qquad D = -\alpha^2. \tag{74}$$

Due to the symmetry of cn the sign of B does not matter and so we choose the positive. Thus we obtain the following solution to the equations of motion (59),

$$\begin{aligned} \dot{x}^0 &= 1 + 2\alpha^2 \,\mathrm{sn}^2(\omega\tau, -\alpha^2), \\ \dot{x}^1 &= -2\alpha \,\mathrm{sn}(\omega\tau, -\alpha^2)\,\mathrm{dn}(\omega\tau, -\alpha^2) \\ x^1 &= d\,\mathrm{cn}(\omega\tau, -\alpha^2). \end{aligned} \tag{75}$$

Let us check the first line of this set of equations against the Hamiltonian constraint, $H = 0$,

$$(p_0 + m\omega^2 (x^1)^2/2)^2 = (p_1)^2 + m^2, \tag{76}$$

which in turn implies

$$\begin{aligned} \dot{x}^0 = \sqrt{1 + (\dot{x}^1)^2} &= \sqrt{1 + 4\alpha^2 \,\mathrm{sn}^2(\omega\tau, -\alpha^2)\,\mathrm{dn}^2(\omega\tau, -\alpha^2)} \\ &= \sqrt{1 + 4\alpha^2 \,\mathrm{sn}^2(\omega\tau, -\alpha^2) + 4\alpha^4 \,\mathrm{sn}^4(\omega\tau, -\alpha^2)} \\ &= 1 + 2\alpha^2 \,\mathrm{sn}^2(\omega\tau, -\alpha^2), \end{aligned} \tag{77}$$

in accord with the first line of eq. (75). Before we visualize the results in eqs. (75), however, we should first investigate the functional relation between proper time τ and coordinate time x^0,

$$x^0(\tau) = \int_0^\tau dt\, \dot{x}^0(t) = \tau + \frac{2\alpha^2}{\omega} \int_0^{\omega\tau} dt\; \mathrm{sn}^2(t, -\alpha^2). \tag{78}$$

Recalling from eqs. (64) that

$$\mathrm{sn}(t, -\alpha^2) = \sin(F^{-1}(t, -\alpha^2)), \tag{79}$$

we may solve the integral by substituting

$$t = F(\psi, -\alpha^2), \qquad\qquad dt = \frac{d\psi}{\sqrt{1 + \alpha^2 \sin^2\psi}}, \tag{80}$$

which yields

$$\begin{aligned} x^0(\tau) &= \tau + \frac{2}{\omega} \int_0^{F^{-1}(\omega\tau, -\alpha^2)} d\psi\, \frac{\alpha^2 \sin^2\psi}{\sqrt{1 + \alpha^2 \sin^2\psi}} \\ &= \tau + \frac{2}{\omega} \int_0^{F^{-1}(\omega\tau, -\alpha^2)} d\psi \left(\sqrt{1 + \alpha^2 \sin^2\psi} - \frac{1}{\sqrt{1 + \alpha^2 \sin^2\psi}} \right). \end{aligned} \tag{81}$$

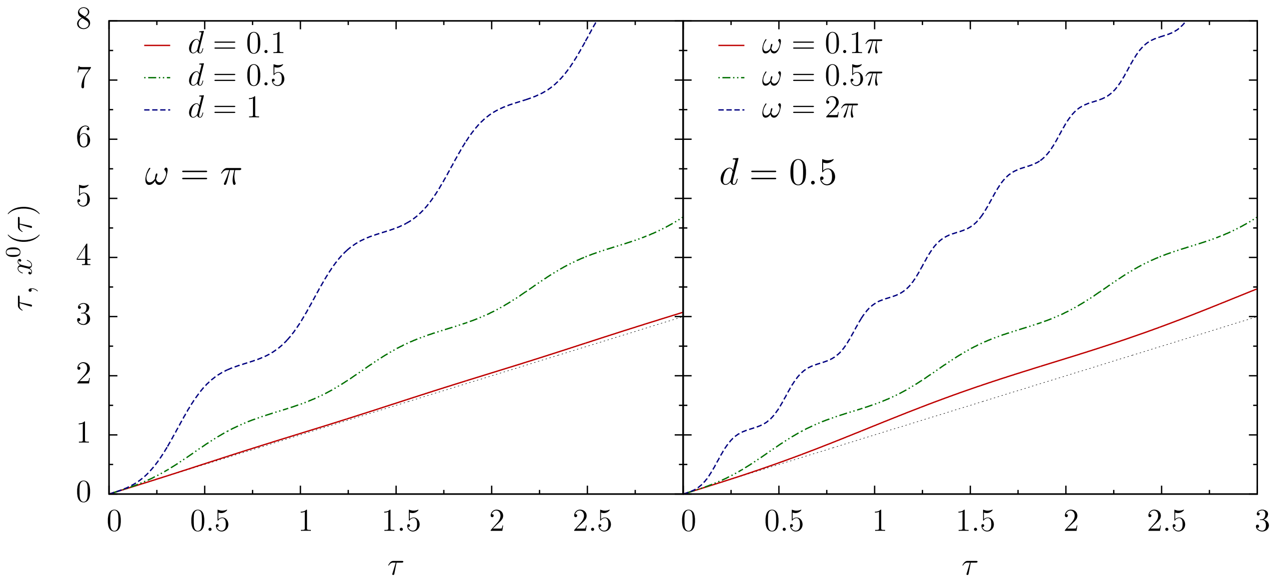

Figure 4: Coordinate time x^0 as a function of proper time τ for different values of d (left) and ω (right) with the other variable held constant, respectively. The proper time τ is also plotted with light dotted lines for comparison.

The first part of the integral represents the elliptic integral of the second kind $E(\phi, \mu)$,

$$E(\phi, \mu) = \int_0^\phi d\psi \sqrt{1 - \mu \sin^2 \psi}, \tag{82}$$

while second part simply resolves to -2τ. Thus we finally obtain

$$x^0(\tau) = -\tau + \frac{2}{\omega} E(\phi(\omega\tau, -\alpha^2), -\alpha^2), \tag{83}$$

with the usual definition of the amplitude ϕ. The solution for $x^0(\tau)$ is plotted in fig. 4 together with the identity for comparison. The influence of the periodic motion on the shape of the curve is quite apparent. One can also nicely see that time runs faster in the inertial frame $\mathcal{C}$ than in the rest-frame of the accelerated particle. Relativistic effects become stronger if α increases, i.e. with increasing d or ω.

Having the twin-paradox in mind it is apparent from fig. 4 that less time passes during a period of motion in the rest-frame of the traveling twin than in the one of the resting twin. As in section 3 we would like to compare the amount of proper time $\Delta\tau$ to the amount of coordinate time Δx^0 that has passed within one period. The Jacobi elliptic functions have a period of $4K$ (along the real axis) and hence we have

$$\frac{\Delta\tau}{\Delta x^0} = \frac{1}{2E(-\alpha^2)/K(-\alpha^2) - 1}, \tag{84}$$

with the complete elliptic integral of the second kind $E(\mu) = E(\pi/2, \mu)$, which satisfies the relation

$$E(-\mu) = \sqrt{1+\mu}\, E\left(\frac{\mu}{1+\mu}\right). \tag{85}$$

Together with eq. (55) we may express the ratio as

$$\frac{\Delta\tau}{\Delta x^0} = \frac{1}{2(1+\alpha^2)\frac{E(\alpha^2/(1+\alpha^2))}{K(\alpha^2/(1+\alpha^2))} - 1}. \tag{86}$$

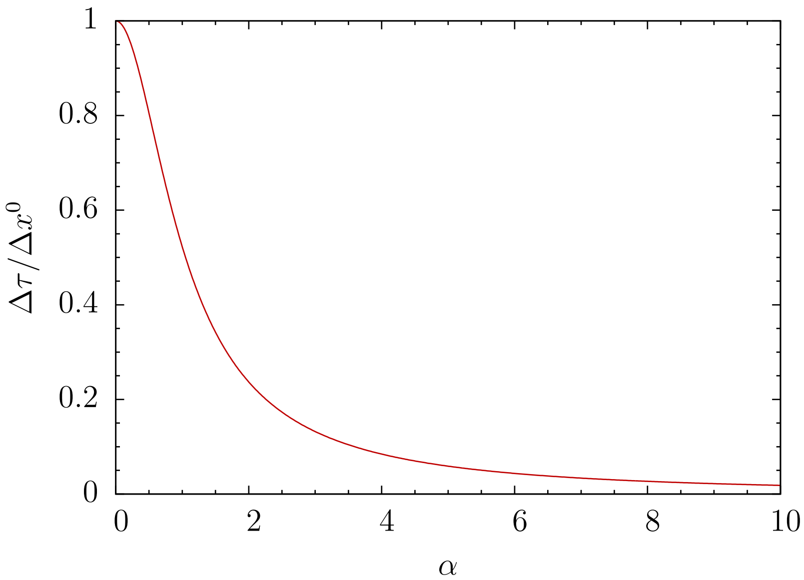

Figure 5: Ratio of the amount of proper time $\Delta\tau$ and coordinate time Δx^0 that has passed between two passages of the particle through $x^1 = d$.

This results is plotted in fig. (5) as a function of α. As in section 3 it is equal to one for $\alpha = 0$ and approaches 0 for $\alpha \to \infty$,

$$1 \leq \frac{\Delta\tau}{\Delta x^0} < 0. \tag{87}$$

Correspondingly, less time has passed between two passages of the accelerated particle (traveling twin) through $x^1 = d$ in its rest-frame than in the coordinate frame $\mathcal{C}$.

To make the difference in the time variables more vivid we have plotted the velocity component $v^1 = \dot{x}^1/\dot{x}^0$ in fig. 6 both as a function of coordinate time x^0 and proper time τ. One can nicely see that the time dilation of x^0, as compared to τ, leads to a longer oscillation period with a strongly stretched curve when $|v^1(x^0)|$ is close to one.

Finally we have plotted the elongation x^1 both as a function of τ and x^0 in fig. 7. The curve shows a similar behavior as in fig. 2 but the comparison of $x^1(x^0)$ with $x^1(\tau)$ shows that non-harmonic motion due to relativistic effects is more pronounced in the inertial frame $\mathcal{C}$.

5 Summary

We have presented the formalism to describe the mechanics of an accelerated, relativistic point particle under the influence of an external four-vector potential A. We have seen that the formalism is actually quite simple and largely analogous to the non-relativistic formalism. Even the general relativistic formalism has only a slightly different form. The detailed calculations, though, are a bit more complicated in the relativistic case. Despite the fact that both examples in this script are not quite realistic due to the neglect of radiative energy-loss, they are perfectly suited to illustrate the relativistic formalism as they are not too simple and can yet be solved analytically. Also they are able to show how the periodic motion is effected by the limiting speed of light and time-dilation in the relativistic regime. Due to the periodic motion both examples also represent a vivid version of the twin-paradox.

I hope this script was helpful to you to learn how to do calculations in relativistic mechanics and to visualize some of the features of relativistic, accelerated motion.

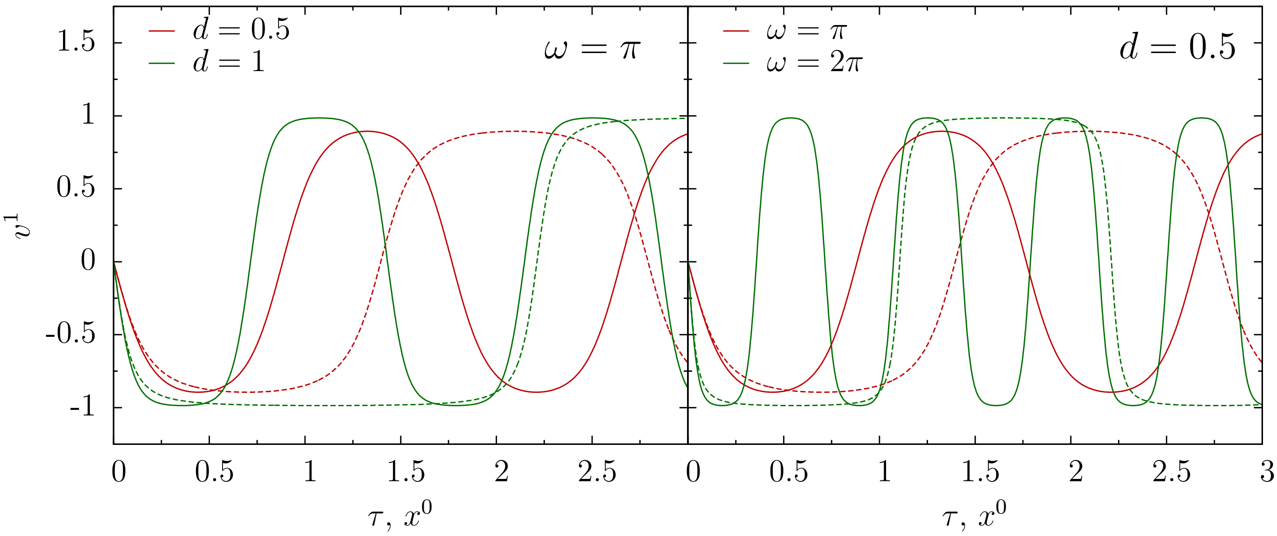

Figure 6: Velocity component $v^1 = \dot{x}^1/\dot{x}^0$ as a function of proper time τ (solid lines) and coordinate time x^0 (dashed lines) for different values of d (left) and ω (right) with the other variable held constant, respectively.

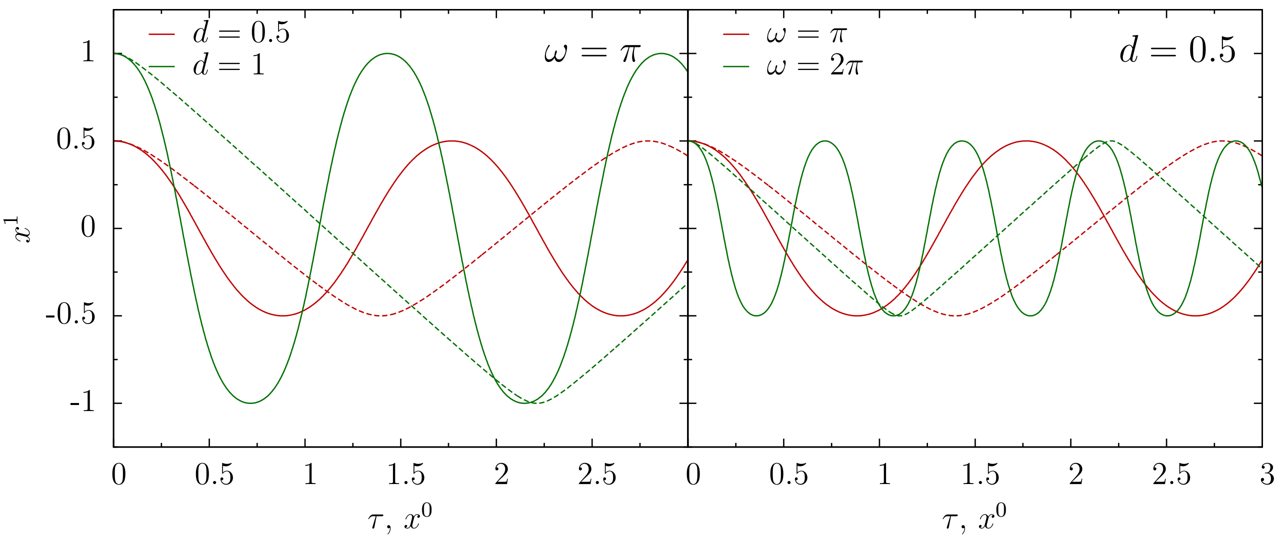

Figure 7: Elongation x^1 as a function of proper time τ (solid lines) and coordinate time x^0 (dashed lines) for different values of d (left) and ω (right) with the other variable held constant, respectively.